Conformal Field Theories Beyond Two Dimensions

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Abstract

I introduce higher dimensional conformal field theory (CFT) for a mathematical audience. The familiar 2D concepts of central charge, local fields, and conformal dimension still play their characteristic roles, but the algebra of symmetries shrinks considerably from the 2D Virasoro to $\mathfrak{so}(d+1,1)$. This makes the exact solution and classification of such CFTs significantly more difficult. I will expand on recent progress that has been made in this direction, known as the conformal bootstrap. I will also elaborate on how higher-dimensional CFT generalizes the work of Belavin, Polyakov and Zamolodchikov of solving statistical models in 2D using the Virasoro algebra to a higher-dimensional setting. Time permitting, I will introduce new results on bounding the conformal dimensions of the 3D supersymmetric Ising model CFT.

1 Motivation

Why study CFTs in > 2 dimensions?

- Belavin, Polyakov + Zamolodchikov 1984
  "infinite conformal symmetry in two dimensions"
- Ising CFTs coming from $V_{c,h}$, minimal model $M(3,4)$
- Key point: Verma modules give insight into critical phenomena of a variety of statistical systems (SLE)
- 2D Ising model: Exactly Solvable $c = 1/2, h_\sigma = 1/16 \to \Delta_\sigma = h_\sigma + \bar{h_\sigma} = 1/8, h_\epsilon = 1/2 \to \Delta_\epsilon = 1$
- 4D Ising model: Exactly Solvable using mean-field methods + renormalization: $\Delta_\sigma = 1, \Delta_\epsilon = 2$
- 3D Ising model: Famously insolvable. New method **Conformal bootstrap** gives results consistent with approximation methods but several orders of magnitude more precise: $\Delta_\sigma = 0.5181489(10), \Delta_\epsilon = 1.412625(10)$ $(c \approx 0.98)$
- Bootstrap also works in 2D case
- *Maldacena Duality* with gravity in hyperbolic spaces

2 What is CFT in Higher Dimensions?

Definition 1. A **conformal field theory** in d dimensions is characterized by the following:

- The conformal group $\mathfrak{so}(d,1)$ with generators (with comparison to 2D case)
  - Dilation, $D \leftrightarrow L_0 + \bar{L}_0$
  - Rotation $M_{\mu\nu} \leftrightarrow L_0 - \bar{L}_0$
  - Translation $P_\mu \leftrightarrow L_i, \bar{L}_i, i \in \mathbb{Z}_+$
  - Special Conformal $K_\mu \leftrightarrow L_{-i}, \bar{L}_{-i}, i \in \mathbb{Z}_+$
- A Hilbert space $\mathcal{H}$ of states
- Operators $\mathcal{O}(x) : \mathbb{R}^d \to \mathrm{End}(\mathcal{H})$ transforming under representations of the conformal group.

  e.g. $\mathcal{O}(0)$ is a rep. of $\mathrm{SO}(d)$, and P_μ acts by $e^{[P_\mu,\cdot]}\mathcal{O}(0) = \mathcal{O}(x)$
- A distinguished vacuum vector, $|0\rangle$
- A set of **primary fields** defined by having $K_\mu \mathcal{O} = 0$. This also implies
  - $D\mathcal{O} = \Delta_{\mathcal{O}}\mathcal{O}$
  - P_μ raises
  - K_μ lowers

  Note in 2D case these would only be called *quasi*-primaries

Proposition 2. *Any local operator is a combination of primaries and descendants For a given primary: it together with its descendants is a **conformal multiplet**.*

- No infinite conformal symmetry $\Rightarrow$ ☹
- *But* Conformal group acts transitively on triples of points in any dimension $\Rightarrow$ ☺

The goal of a field theory is to obtain explicit expressions for all correlation functions:

$$\langle 0|\mathcal{T}\left\{\mathcal{O}_1(\mathbf{x}_1)\dots\mathcal{O}_k(\mathbf{x}_n)\right\}|0\rangle$$

Make note about time ordering
Conformal invariance helps us in this task.
Two point functions:

$$\langle\mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\rangle = \frac{C}{|\mathbf{x}_1 - \mathbf{x}_2|^{2\Delta_{\mathcal{O}}}} \tag{1}$$

Three point functions:

$$\langle\mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_3(\mathbf{x}_3)\rangle = \frac{f_{123}}{x_{12}^a x_{23}^b x_{34}^c} \tag{2}$$

For scaling dimensions Δ to match, we require $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$. In fact:

$$a = \Delta_1 + \Delta_2 - \Delta_3, \text{ etc.}$$

Four point functions are now more difficult, an in general may depend on the **cross-ratios** (conformally invariant combination of the x_i):

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad u = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

Four point functions can depend nontrivially on the cross ratios:

$$\langle \mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_3(\mathbf{x}_3)\mathcal{O}_4(\mathbf{x}_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \tag{3}$$

3 The Operator Products Expansion, Crossing Symmetry, and the Conformal Bootstrap

As before there is a **state-operator correspondence**

$$\mathcal{O}(0) \leftrightarrow |\mathcal{O}\rangle := \mathcal{O}(0)\,|0\rangle$$

Here we take the state $|\mathcal{O}\rangle$ to $\mathcal{O}(x)$. Note correspondence with 2D case where we define

$$Y(\cdot, z) : V \to \mathrm{End}(V)$$

A primary operator gives rise to a primary state, a lowest-weight module for $\mathfrak{so}(d,1)$

Now we can write:

$$\mathcal{O}_i(x)\mathcal{O}_j\,|0\rangle = \sum_k C_{ijk}(x, \partial_\mu)\mathcal{O}_k(0)\,|0\rangle \tag{4}$$

where C_{ijk} is a function of x, ∂. Indeed it can be shown that C_{ijk} is proportional to the constant f_{ijk} times a (known) differential operator depending on only on the Δ values.

Corollary 3. *All correlation functions are determined by the scaling dimensions Δ_i in the theory, and the OPE coefficients f_{ijk}*

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \sum_k C_{12k}(x_{12}, \partial_2)\,\langle \mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle \tag{5}$$

Do this recursively until 1-points functions, and we have $\langle \mathcal{O}\rangle = 0$ except for the identity, which has $\langle 1\rangle = 1$

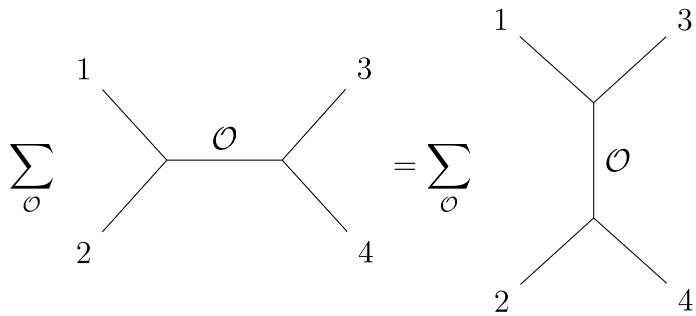
Now to conclude, let's look at 4-point functions. Let's restrict to identical scalar correlators

$$\begin{aligned}
\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \sum_{k,k'} f_{\phi\phi k} f_{\phi\phi k'} C_{\phi\phi\mathcal{O},a}(x_{12},\partial_2) C_{\phi\phi\mathcal{O}',b}(x_{34},\partial_4)\,\langle\mathcal{O}(x_2)\mathcal{O}'(x_4)\rangle \\
&= \sum_k f^2_{\phi\phi k} C_{\phi\phi\mathcal{O}}(x_{12},\partial_2) C_{\phi\phi\mathcal{O}',b}(x_{34},\partial_4)\frac{I^{ab}}{|x_{24}|^{2\Delta_\mathcal{O}}} \\
&= \sum_k f^2_{\phi\phi k} g_{\Delta_\mathcal{O},\ell_\mathcal{O}}(u,v)
\end{aligned}$$

where we have defined $g_{\Delta_\mathcal{O},\ell_\mathcal{O}}(u,v)$ to satisfy this. This is our **conformal block decomposition**

Conformal blocks are known explicitly in *even dimensions* using techniques involving conformal Casimir. Only series expansions through recursion of coefficients are known in odd dimensions. Its not obvious that they depend on only cross ratios

In general the principle is this:



Because this should be invariant under permutation, we get two constraints on g

$$g(u,v) = g(u/v, 1/v), \qquad g(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} g(v,u) \tag{6}$$

This last condition then becomes:

$$\sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}}(v^{\Delta_\phi} g_{\Delta,\ell}(u,v) - u^{\Delta_\phi} g_{\Delta,\ell}(v,u)) = 0$$

But in a unitary (reflection-symmetric) CFT, we have that $f_{\phi\phi\mathcal{O}}$ are real, and their squares are thus positive.

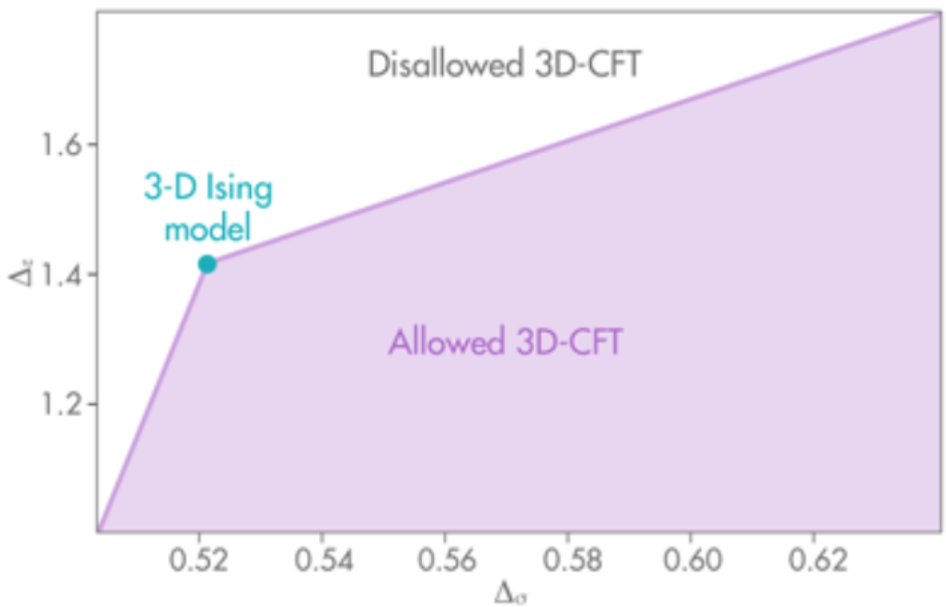
The more complicated term in parentheses, when expanded in a polynomial in $z, \bar{z}$ around some point to a finite order $\partial_z^n \partial_{\bar{z}}^m, m+n=\Lambda$ becomes just a finite-dimensional vector of polynomials depending on just $\Delta_\phi, \Delta_\mathcal{O}$ and $\ell_\mathcal{O}$ (this vector has one component each $\mathcal{O}$). We can thus write this (finite dimensionally!) as:

$$\sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} F_{\Delta,\ell}(z,\bar{z}) = 0 \tag{7}$$

Concept 4 (Conformal Bootstrap)**.** *If there exists a function α acting on the space of polynomial vectors F such that that $\alpha(F_i) > 0$ for each component i, then crossing symmetry is violated, and the given data does not represent a valid CFT.*

4 Example: Regions in the Space of 3D Ising-like CFTs

Using mixed correlators $\langle \sigma\sigma\epsilon\epsilon \rangle$, we get this island in the space of 3D CFTs on two relevant operators:



There are much stronger bounds for this island using further known constraints on crossing symmetries of three-point functions in such theories, together with a technique of scanning over ratios of three-point coefficients, known as **theta-scan**. We have extended this to the larger space of CFTs away from this island. Moreover we have novel bounds on a related CFT in this space known as the **supersymmetric Ising model**.